

APPLICATIONS of METHODS
of
TWO-DIMENSIONAL QUANTUM FIELD THEORY
to
PROBLEMS
in
ELEMENTARY PARTICLE PHYSICS AND
STATISTICAL MECHANICS

Inaugural - Dissertation
zur Erlangung der Doktorwürde
des Fachbereichs Physik
der Freien Universität Berlin

vorgelegt von
Bernd Berg
aus Delmenhorst

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Preface

This thesis deals with physics in $D = 2$ (one space and one time or two space and no time) dimensions. Particle scattering, in the massive Thirring model is treated in perturbation theory up to the one loop approximation (part I), and the relationship between the Baxter model and scale invariant field theory is investigated (part II).

The Thirring model is a two-dimensional field theory of Fermions, selfinteracting via a quadrilinear coupling term. The Baxter (eight vertex) model belongs to the ferroelectric models of statistical mechanics in two space dimensions. The unifying aspect of such apparently different topics comes in through the treatment of the Baxter model with methods of two-dimensional quantum field theory.

In part I of this work conventional perturbation theory is used to check results which provide a basis for the derivation of the S-matrix of the massive Thirring model. There is some hope that even the Wightman functions of the massive Thirring model may be constructed in the not too far future. For the massless case the Wightman functions are known.

We treat the Baxter model in part II. The correlation functions of the Baxter model have been conjectured to be related to the Wightman functions of an underlying field theory by analytic continuation from the Minkowskian to the Euclidean region. Although we are not able to construct the underlying field theory explicitly, nevertheless the methods which were applied successfully for solving the massless Thirring model enable the construction of the Baxter model correlation functions on a straight line.

We finish the preface with a few general remarks about two-dimensional physics.

The simplicity of restricting motion to one spatial dimension is obvious. Much of the complexity of the real world comes from the fact that particles can get around each other. Therefore we are often able to obtain in two dimensions analytic solutions of involved problems which cannot be solved in the realistic space-time. These solutions, naturally, have lost much of the richness of the real world and they are not always free of pathologies, but on the other hand, they have frequently turned out to be complex enough to yield interesting physical features which were sometimes lost by using oversimplified methods in the realistic space-time.

In elementary particle physics the investigation of the (massless) Thirring model has played an essential role in the development of Wilson's short-distance and light-cone expansions which incorporate the concept of anomalous dimensions. The Wilson expansions in turn had interesting physical applications to deep inelastic lepton and hadron scattering. (For a review see e.g.: Y. Frishman, Phys. Rep. 13 (1974) 1) Other interesting results are known from the $SU(n)$ -Thirring model, from the Schwinger model, and even from the Federbush model.

The two-dimensional ferromagnetic (Ising) and ferroelectric (six, eight vertex) models in solid state physics are now known to be related to the backward scattering model which in turn is related to the Thirring model. In statistical mechanics the two-dimensional Ising model is the most well-known and perhaps most significant example of an explicitly solved model. The validity of the Van der Waals and mean field theories of phase transitions was doubted since its solution by Onsager in 1944, and this date may be considered as the starting point for the modern era of phase transitions. Other two-dimensional models in statistical mechanics provided a serious testing ground for approximate mathematical methods. Finally occasions arise of direct physical significance through applications to chain molecules. (For reviews with respect to solid state physics see e.g.: C. Domb and M.S. Green, "Phase Transitions and Critical Phenomena", Academic Press, especially Vol I; E.H. Lieb and D.C. Mattis, "Mathematical Physics in One Dimension", Academic Press 1966)

PART ONE

The Massive Thirring Model:

Particle Scattering in Perturbation Theory

THE MASSIVE THIRRING MODEL:
PARTICLE SCATTERING IN PERTURBATION THEORY

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Abstract:

We compute $(n \rightarrow m)$ -particle, antiparticle scattering in the massive Thirring model up to the one loop approximation. In agreement with presumed higher order conservation laws there is no particle production and the S-matrix factorizes in terms of two-particle scattering amplitudes.

I. INTRODUCTION

Since Coleman's equivalence /1/, /2/ of the Sine-Gordon (SG) theory and the massive Thirring model (MTM) there have been numerous investigations of both models. Here we first wish to review briefly the development.

At the time of Coleman's investigation many interesting properties of the SG theory were already known, however, similar aspects of the MTM had not been studied. On the classical level the SG theory was completely solved by the inverse scattering method /3/. As a consequence, infinitely many higher conserved currents (corresponding to powers of the momenta $\sum_k (p_k)^n$ ($n = 3, 5, \dots$)) were known for the classical SG theory /4/. Later these currents were shown to survive quantization /5/, /6/. The first evidence for this fact came from the previously known computation of the $(n \rightarrow m)$ -particle scattering amplitude of the SG model up to the one loop approximation /7/ in which the following results were obtained: a) the number of particles remains unchanged, b) the sets of incoming and outgoing momenta are equal, c) the S-matrix factorizes (in the considered approximation) in terms of $(2 \rightarrow 2)$ -particle scattering amplitudes.

The quasiclassical approximation of the SG theory /8/, /9/ exhibited the existence of soliton and antisoliton states, and the original "elementary particle" of the theory was reobtained as one in a series of bound states. In this approximation the conservation laws mentioned above turned out to be true also for the scattering of solitons, antisolitons and bound states /9/. A variety of plausibility arguments /8/, /9/, /10/, /11/ suggests the semiclassical results to be exact also for the quantized SG model.

The soliton of the SG theory corresponds (by Coleman's equivalence) to the fundamental Fermion in the MTM. Therefore, the (quasiclassical) SG results concerning the spectrum and soliton scattering are true in the MTM to the extent of the validity of the hypothesis that the quasiclassical approximation of the

SG theory is exact. The challenge is to establish the assumed properties exactly. Various results were obtained in this direction. For example the spectrum has been reobtained for a lattice version of the MTM /10/, as well as by other methods /11/. The exact nature of the continuum limit of the lattice theory is, however, unclear.

The question of the higher order conservation laws in the MTM has been investigated by several authors. As a first step the vanishing of particle production ($2 \rightarrow 4$) was demonstrated in the tree /12/ and one loop /13/ approximation. Then for the classical MTM (i.e. the tree approximation) higher order conservation laws were obtained /5/, /14/.

For the quantized MTM so far only the first nontrivial higher current has been rigorously (more precisely: in all orders perturbation theory) treated /15/. This conservation law implies: The sets of outgoing and incoming momenta are equal in $(3 \rightarrow 3)$ -particle scattering, particle production ($2 \rightarrow 2n$) is forbidden, and the $(3 \rightarrow 3)$ -particle S-matrix factorizes /16/ in terms of two particle scattering. The results are commonly believed to remain true for $(n \rightarrow m)$ -particle scattering. In this paper we support this belief by an explicit calculation up to the one loop approximation.

Our result (section II. c) and III. b)) has its parallel in that of Araf'eva and Korepin /7/ for the SG theory. (Remember, however, the "elementary particle" of the SG theory does not correspond to the fundamental Fermion in the MTM.) We find it worthwhile to describe the computational method in some detail because any (convergent) two-dimensional loop integral may be tackled by similar methods and is then obtained in terms of tree graphs.

P. Weisz /17/ has computed the $(2 \rightarrow 2)$ -particle amplitude of the MTM up to the two loop approximation and found factorization of the two loop integrals in terms of one loop integrals. Quite generally, we conjecture that in the MTM and in the SG theory n -loop integrals reduce to $(n-1)$ -loop integrals and that the

perturbation series for the S-matrix may be summed up explicitly. This program seems to us at least within the reach of possibilities as in a remarkable recent article Zamolodchikov^{*}/18/ has constructed a $(2 \rightarrow 2)$ -particle S-matrix which is nearly certainly exact. His construction is based on several assumptions which include the exactness of the semiclassical spectrum. The final result is a solution of the unitarity and crossing relations which matches with perturbative calculations (in contrary to a previous conjecture /19/ which failed already in first and second order /20/) up to the third order /17/.

Our material is organized as follows: In section II.a) some notations and preliminaries are summarized, section II.b) contains the (technical) main statement of the tree approximation, the vanishing of the contributions of the principal values of the propagators. Section III.a) treats the general one loop integral. The sections II.b) and III.a) are somewhat technical, the main results are given in section II.c) (tree approximation) and III.b) (loop approximation). Finally, we note, although all computations of this paper can be easily done by hand, the author found it convenient to do some calculations with Veltman's algebraic computer program Schoonship /21/.

^{*Note added in proof:}

Meanwhile, relying on Zamolodchikov's solution, the soliton form factors have been computed /24/ and a self-consistent S-matrix of all sectors (including bound states) has been derived /25/.

II. $(n \rightarrow m)$ -PARTICLE SCATTERING IN TREE APPROXIMATION

a) Notation and Preliminaries

We use the γ -matrices $\gamma^\mu = \sigma_\mu$, $\gamma^5 = \lambda \sigma_4$ and choose the spinors to be of the form

$$u(p) \equiv u(a) = \sqrt{m} \begin{pmatrix} 1 \\ a \end{pmatrix}, \quad v(p) \equiv v(a) = \sqrt{m} \begin{pmatrix} 1 \\ -a \end{pmatrix} \quad (2.1)$$

where a is defined by the light-cone variables on the mass-shell

$$p^\pm = p^0 \pm p^1 = m a^{\pm 1}.$$

Our spinors are normalised such that

$$\frac{1}{a} u \bar{u} = \not{p} + m, \quad \frac{1}{a} v \bar{v} = \not{p} - m \quad (2.2)$$

The replacement of an incoming (outgoing) particle by an outgoing (incoming) antiparticle corresponds to the substitution law

$$a \rightarrow -\hat{a}$$

$$u(a) \rightarrow u(-\hat{a}) = v(\hat{a}), \quad \bar{u}(a) \rightarrow \bar{u}(-\hat{a}) = \bar{v}(\hat{a}) \quad (2.3)$$

Here the variables a , $\hat{a}$ correspond to particles, antiparticles respectively.

We are now interested in $(n \rightarrow n)$ -particle scattering. The connected part of the $(n \rightarrow n)$ -particle scattering matrix element is given by

$$M_{nn}^{\text{conn}} = \text{out} \langle p_{1n} \dots p_{n+n} | p_1 \dots p_n \rangle_{\text{in}}^{\text{conn}} \\ = (2\pi)^2 \delta^{(4)}(p_1 + \dots + p_n - p_{n+1} - \dots - p_{2n}) \left(\frac{1}{i\epsilon} \right)^n \frac{A_{nn}}{(a_1 \dots a_n)^{1/2}} \quad (2.4)$$

All variables in (2.4) are taken on mass-shell. Additionally, energy-momentum conservation gives the two relations

$$a_1^{\pm 1} + \dots + a_n^{\pm 1} = a_{n+1}^{\pm 1} + \dots + a_{2n}^{\pm 1} \quad (2.5)$$

In the approximation considered in this section A_{nn} is given to be the sum of all configurations of tree graphs with n incoming and n outgoing momenta.

$$A_{nn}^{\text{tree}} = \sum_{\text{graphs}} \alpha_{nn}(t, d') B_{nn}^{\text{tree}}(t, d')$$

The index t refers to the topology. Under a configuration we understand a graph with fixed topology t and fixed assignment $(\alpha, \alpha') = (\pi_1, \dots, \pi_n, \pi'_1, \dots, \pi'_n)$ of external momenta.

For the simplest case of $(2 \rightarrow 2)$ -particle scattering (Fig. 1a) the amplitude is easily seen to be

$$\begin{aligned} A_{11}^{\tau} &= 4 \cdot (-i) \cdot i g \tilde{m}_3 \tilde{m}_4 \varepsilon_{43} \varepsilon_{12} u_1 u_2 \\ &= -2 i g m^2 (a_4 - a_3)(a_2 - a_1) \end{aligned} \quad (2.6)$$

and energy-momentum conservation (1.5) admits only two discrete solutions.

The result of ref. 12 for $(3 \rightarrow 3)$ -particle scattering turns out to be the appropriate first step in an inductive proof of the general result concerning $(n \rightarrow n)$ -particle scattering in the tree approximation. The amplitude is obtained by adding up the permutations (with respect of external momenta) of one graph (Fig. 2). We denote the contribution coming from the principal part of the internal propagators PA_{33}^{τ} and arrive after some calculation at the expression

$$A_{33}^{\tau} = i g^2 8 m^2 (a_6 - a_5)(a_5 - a_4)(a_4 - a_3) \cdot Z$$

$$\text{with } Z = \sum_{i=1}^3 \frac{(a_{\sigma_{i+1}} a_{\sigma_{i+2}} - a_i^2)(a_{\sigma_{i+1}} - a_{\sigma_{i+2}})}{N_i}$$

where

$$N_i = \frac{1}{a_i^2} (a_i + a_{\sigma_{i+1}})(a_{\sigma_{i+2}} + a_i)(a_i - a_6)(a_i - a_5)(a_i - a_4)$$

and

$$\sigma_i = \begin{cases} i & \text{if } i = 1, 2, 3 \\ (i-1) & \text{if } i = 4, 5. \end{cases}$$

Energy-momentum conservation implies

$$N_1 = N_2 = N_3$$

and, therefore,

$$PA_{33}^{\tau} \equiv 0. \quad (2.7)$$

b) The Main Statement of the Tree Approximation

We now generalize the result (2.7) to $(n \rightarrow n)$ -particle scattering. By PA_{nn}^{τ} we denote the contribution which we get from (2.4) to A_{nn}^{τ} by taking from all propagators the principal parts only. The main

result of this section is

$$PA_{nn}^{tr} \equiv 0. \quad (2.8)$$

Proof: We assume the result already being proved for A_{kk}^{tr} ($K \leq n-1$) and proceed by induction.

We take $a_4 \dots a_n$ fixed (complex values admitted) and consider $z = a_1$ as a complex variable. a_2, a_3 are fixed via energy-momentum conservation to be

$$a_3 = \frac{A}{2} (1 + \sqrt{1 - \frac{4}{A^2}})$$

with

$$A = a_{n+1} + \dots + a_{2n} - z - a_4 - \dots - a_n$$

$$B = a_{n+1}^{-1} + \dots + a_{2n}^{-1} - z^{-1} - a_4^{-1} - \dots - a_n^{-1}$$

If the amplitude $A_{nn}^{tr}(z) \equiv w(z)$ is not constant it satisfies an irreducible algebraic equation of the form

$$\sum_{i,k=0}^{r,s} a_{ik} w^i z^k = 0$$

with $r \geq 1$ and $a_{rk} \neq 0$ for at least one k . By a well-known mathematical theorem /22/ this equation determines a compact Riemann surface $\mathcal{T}$ such that w becomes a single valued function $w = w(\xi)$ on $\mathcal{T}$. By a projection $z = F(\xi)$, $\xi \in \mathcal{T}$ the surface $\mathcal{T}$ is a covering surface of the complex plane. The projection can be used to define local uniformizing variables such that for every $\xi \in \mathcal{T}$ (branch points as well as ordinary points) there exists a neighbourhood where w is a single valued function $w = w(t)$ of a local complex uniformizing variable t . On the whole surface the numbers of zeros and poles (counted in the uniformizing variables) are equal. Under certain technical restrictions - they may be summarized as the assumption that in some sense the external momenta are non-degenerate - we show $w(z) \equiv PA_{nn}^{tr}(z)$ has more zeros than poles and is, therefore, identically zero. In a last step we get rid of the restrictions and obtain $PA_{nn}^{tr}(a_1 \dots a_{2n}) \equiv 0$.

Discarding $z = F(\xi) \rightarrow \infty$ for the moment, we are left with poles which come from the propagators. We now assume for any configuration contributing to PA_{nn}^{tr} : 1) No propagator is infinite for all $\xi \in \mathcal{T}$. 2) The poles of different propagators are lying at different points on the Riemann surface. Under these conditions each pole in a configuration is related to exactly one propagator (Fig.3).

The principal part of the propagator can be written

$$\frac{i}{\sum_{i=1}^K \varepsilon_{\pi_i} p_{\pi_i} - m} = \frac{i}{(\sum_{i=1}^K \varepsilon_{\pi_i} p_{\pi_i})^2 - m^2} \sum_{i=1}^K \frac{\varepsilon_{\pi_i}}{a_{\pi_i}} \mu(a_{\pi_i}) \bar{\mu}(a_{\pi_i}) \quad (2.9)$$

with K odd, $\{\pi_1 \dots \pi_K\} \subset \{1, \dots, 2n\}$ and $\varepsilon_{\pi_i} = \begin{cases} +1 & \text{if } \pi_i \in \{1, \dots, n\} \\ -1 & \text{if } \pi_i \in \{n+1, \dots, 2n\} \end{cases}$.

In the neighbourhood of the considered pole we parametrize the momentum flow $q = \sum_{i=1}^K \varepsilon_{\pi_i} p_{\pi_i}$ through the propagator in terms of a local uniformizing parameter t such that $q_{2n} = q_0 = q(t_p)$. The factors on the right-, left-hand-side which are denoted T_L, T_R in Fig. 3 would be configurations of tree type if q were on the mass-shell. This is generally not the case, but it is true in the limit $t \rightarrow t_p$ in which also $\sum_{i=1}^K \frac{\varepsilon_{\pi_i}}{a_{\pi_i}} \mu(q_{\pi_i}) \bar{\mu}(q_{\pi_i}) \rightarrow \mu(q_0) \bar{\mu}(q_0)$. The induction hypothesis implies immediately

$$0 = \lim_{t \rightarrow t_p} [(t - t_p)^{\alpha} T_L(t) \cdot \frac{\sum \varepsilon_{\pi_i} \mu_{\pi_i} \bar{\mu}_{\pi_i}}{q(t)^2 - m^2} \left(\sum_{i=1}^K T_R(t) \right)]$$

Here α denotes the order of the pole counted in the local uniformizing parameter. Then there are no poles at points $\xi \in \mathcal{F}$ which have a finite projection $z = F(\xi)$.

We now treat $w(\xi)$ for $z = F(\xi) \rightarrow \infty$. This is done by power counting. We require additionally 3) $\sum_{i=1}^K \varepsilon_{\pi_i} \tilde{a}_{\pi_i} \neq 0$ for all $\pi \in \{1, \dots, 2n\}$ with $\tilde{a}_i = \left[\sum_{j=1}^K \varepsilon_{\pi_j} a_j^{-1} \right]^{-1}$ and $\tilde{a}_i = a_i$ ($i = \pi_1, \dots, \pi_K$).

Under this restriction $z \rightarrow \infty$ implies $a_i \rightarrow \infty$ (linearly) and $\tilde{a}_i \rightarrow \text{const.}$ or vice versa (depending on the sheet of $\mathcal{F}$).

Using (2.9) any propagator is seen to stay finite for $z \rightarrow \infty$.

The general term contributing to PA_{n-1}^{Fey} is of a form proportional to

$$\mu_{\pi_1} \dots \mu_{\pi_n} \bar{\mu}_{\pi_{n+1}} \dots \bar{\mu}_{\pi_{2n}} \prod_{i=1}^n \frac{1}{\sum_{j=1}^n \varepsilon_{\pi_j} - m} \text{ with } \sum_{j=1}^n \varepsilon_{\pi_j} = \sum_{j=1}^n \varepsilon_{\pi_j}, \pi_j \in \{1, \dots, 2n\}$$

with the spinors given by (2.1). Hence, a behaviour $w(\xi) \sim z^n$ for $z = F(\xi) \rightarrow \infty$ is the worst possible.

By Fermi statistics $PA_{n-1}^{\text{Fey}}(\xi)$ has at least $(n-1)$ zeros. It is therefore identically zero for $n \geq 4$.

It remains to discuss the assumptions. Requirement 1) is easily reformulated as the algebraic condition

$$\left(\sum_{i=1}^K \varepsilon_{\pi_i} a_i \right)^{-1} \neq \sum_{i=1}^K \varepsilon_{\pi_i} a_i^{-1} \text{ for any } \pi \in \{1, \dots, 2n\}.$$

Also 2) can be seen to amount to finitely many algebraic inequalities. The meaning of these algebraic inequalities is, that in the $(2n-3)$ -dimensional parameter space $(a_1 \dots a_{2n})$ the amplitude is only up to finitely many hyperplanes known to be zero. The algebraic character of $PA_{nn}^{\sim}$ in all parameters ensures that we can easily show $PA_{nn}^{\sim}$ to be identically zero.

c) Results in the Tree Approximation

An immediate consequence of the main result is, that only δ -function parts of the propagators contribute to the scattering amplitude $A_{nn}^{\sim}$. This enables us to express $(n \rightarrow n)$ -particle scattering very simply in terms of $(2 \rightarrow 2)$ -particle scattering. Using (momenta are on shell)

$$\delta^{(n)}(p_1 + p_2 - p_3 - p_4) = \frac{2 p_1^0 p_2^0}{\sqrt{-t_{12}} s_{12}} \left\{ \delta^{(n)}(p_1^2 - p_2^2) \delta^{(n)}(p_3^2 - p_4^2) + (2n+1) \right\} \quad (2.10)$$

we rewrite $(2 \rightarrow 2)$ -particle scattering in the convenient form

$$out \langle p_4 p_3 | p_1 p_2 \rangle^{in} = in \langle p_4 p_3 | p_1 p_2 \rangle^{in} \frac{i 2 \delta_{12}(1,2)}{s_{12}} \quad (2.11a)$$

With the Mandelstam variables $s_{12} = (p_1 + p_2)^2$, $t_{12} = (p_1 - p_2)^2$ the two particle scattering phase is in the tree approximation

$$\delta_{12}(1,2) = -\frac{i}{2} \gamma \left(-t_{12} / s_{12} \right)^{1/2} + O(\gamma^2) \quad (2.11b)$$

We now consider the $(n \rightarrow n)$ -particle scattering matrix element

$out \langle p_{1n} \dots p_{nn} | p_1 \dots p_n \rangle^{in}$
Energy-momentum conservation at each vertex together with all (external and internal) momenta known on mass-shell admits at each vertex only the two discrete solutions of $(2 \rightarrow 2)$ -particle scattering for the outgoing momenta. As a consequence the sets of incoming and outgoing momenta are equal:

$$\{p_1 \dots p_n\} = \{p_{1n} \dots p_{nn}\}$$

To derive an explicit expression for the matrix element we first note that the δ -function at an arbitrary vertex v is

$$\delta^{(n)}((p_{v1} + p_{v2} - p_{v3})^2 - m^2) = \frac{p_{v3}^0}{\sqrt{-t_{v1v2}} s_{v1v2}} \left\{ \delta^{(n)}(p_{v1}^2 - p_{v2}^2) + (v_1 + v_2) \right\} \quad (2.12)$$

There are $(n-2)$ such internal δ -functions which reduce the overall energy momentum $\delta^{(4)}$ -function to a function of two incoming and two outgoing momenta. Applying (2.10), (2.12) each configuration becomes proportional to a sum of terms of the form

$$2p_{\pi_1}^0 \delta^{(4)}(p_{\pi_1}^0 - p_{\pi_2}^0) \dots 2p_{\pi_{n-2}}^0 \delta^{(4)}(p_{\pi_{n-2}}^0 - p_{\pi_{n-1}}^0) \lambda_1 \delta_{\lambda_1}(\lambda_1, \kappa_1) \dots \lambda_2 \delta_{\lambda_2}(\lambda_2, \kappa_2)$$

with π a permutation of $1 \dots n$ and $\lambda_v, \kappa_v \in \{1 \dots n\}$, $\lambda_v, \kappa_v, (\lambda_v, \kappa_v) \neq (\lambda_{v'}, \kappa_{v'})$ $\forall v \neq v'$

Symmetry with respect to the incoming momenta requires that all such terms are equally weighted after doing the sum over all configurations. Including now disconnected (up to order g^{n-1}) graphs the summed up tree approximation for $(n \rightarrow n)$ -particle scattering becomes

$$\langle p_{1n} \dots p_{n+1} | p_1 \dots p_n \rangle_{\text{out}}^{\text{in}} = \langle p_{1n} \dots p_{n+1} | p_1 \dots p_n \rangle_{\text{in}} \prod_{\lambda=1}^n \lambda^2 \delta_{\lambda}(\lambda, \eta) \quad (2.13)$$

with $\delta_{\lambda}(\lambda, \eta)$ given by (2.11b) and terms of order higher than g^n have to be neglected.

$(n \rightarrow n)$ -antiparticle scattering can be treated along the same lines. The two-antiparticle scattering phase $\delta_{\lambda\lambda}$ is obtained from the two-particle scattering phase by applying the substitution rule (2.3).

A scattering process including particles as well as antiparticles leads to vertices of (particle, antiparticle) $\rightarrow$ (particle, antiparticle) type (Fig. 1b). The corresponding phase $\delta_{\lambda\lambda}(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4)$ is obtained by the replacement $\alpha_1 \rightarrow -\hat{\alpha}_1$, $\alpha_2 \rightarrow -\hat{\alpha}_2$ and yields a contribution proportional to $(S_{\lambda\lambda} / -i\lambda_{\lambda})^{1/2}$ for forward (i.e. $\hat{p}_1 = \hat{p}_1$, $p_1 = p_1$) and proportional to $m^2 (1 / -i\lambda_{\lambda} S_{\lambda\lambda})^{1/2}$ for backward (i.e. $\hat{p}_1 = p_1$, $\hat{p}_2 = p_2$) scattering. The matrix element factorizes again:

$$\langle \hat{p}_{1n} \dots \hat{p}_{n+1} | \hat{p}_1 \dots \hat{p}_n \rangle_{\text{out}}^{\text{in}} = \langle \hat{p}_{1n} \dots \hat{p}_{n+1} | \prod_{\lambda=1}^n (S_{\lambda\lambda}) | \hat{p}_1 \dots \hat{p}_n \rangle_{\text{in}} \quad (2.14)$$

Here the $\hat{p}_j$ are particles p_j or antiparticles $\hat{p}_j$ and we have adopted the "canonical" ordering prescription $|25/ p_1 \dots p_n$.

As a last point the question of pair-production is easily settled. Analyticity connects the amplitudes in question with the considered ones. As the contributions involving principal parts of the propagators are identically zero on the whole Riemann surface, we have not to bother about the correct branch and any contribution may again only come from the δ -function part. Therefore, pair creation is forbidden at any vertex and, consequently, at all.

III. (n→m)-PARTICLE SCATTERING IN THE ONE LOOP APPROXIMATION

a) The One Loop Integral

The general one loop integral (Fig.4) in the massive Thirring model is a tensor product

$$L_m \alpha_1 \beta_1 \dots \alpha_n \beta_n = \int dK \prod_{i=1}^m S_F(\alpha_i, K_i)_{\alpha_i \beta_i} \quad \text{with } dK = dK^0 dK^1 \quad (3.1)$$

and

$$S_F(K_i) = i \frac{K_i + \not{P}_i + m}{(K_i + P_i)^2 - m^2 + i\epsilon}$$

$\alpha_i = \pm 1$ refers to the charge flow. The tensor indices ($\alpha_i = 1, 2$; $\beta_i = 1, 2$) are suppressed in most of the following formulae. We first restrict ourselves to the case $n \geq 3$. Closing the integration contour in the lower plane we carry out the K_0 -integration with the residue theorem and obtain

$$L_m = -2\pi i \int dK^1 \sum_{i=1}^m \frac{\epsilon_i K_i^1 + m}{2\omega_i} / \prod_{i=1}^m \frac{\epsilon_i K_i^1 + m}{2\omega_i P_{i-1}^0 - 2K_i^1 P_{i-1}^1 + P_{i-1}^2 + i\epsilon} \quad (3.2)$$

with

$$\omega_i = \sqrt{(K_i^1)^2 + m^2}, \quad P_{i-1}^0 = P_i^0 - P_i^1$$

For $K^1 \rightarrow \pm\infty$ the integral seems to diverge linearly if we take only K^+ -contributions from the numerator. This would be in contradiction to formal power counting. The detailed calculation exhibits: the integral decreases faster than $1/K^1$ for $K^1 \rightarrow \pm\infty$ as was expected by power counting. For example, the constant tensor component

$$\prod_{i=1}^m (\delta_{\alpha_i \alpha_i} \delta_{\beta_i \beta_i}) \cdot \frac{1}{2} \sum_{i=1}^m \prod_{j \neq i} \frac{1}{P_{j-1}^2}$$

which is obtained for $K^1 \rightarrow \pm\infty$ is shown to be zero (non degenerate momenta assumed in the sense of chapter II) by observing that the poles in the difference variables vanish. In the same way (slightly more tedious) the coefficients of the terms proportional to $1/K^1$ are seen to disappear.

The integral (3.2) is identical with $\lim_{K^1 \rightarrow \infty} \left\{ -m \sum_{i=1}^m \left(\int_{-K^1 P_{i-1}^0}^0 + \int_0^{K^1 P_{i-1}^0} \right) dK_i^1 \right\}$.

Each integrand in this sum becomes rational by the substitution
/24/

$$X_i = \frac{K_i^1}{W_i - m}$$

A single term is

$$\tilde{J}_i(X_i) = (X_i^2 - 1)^{-1/2} (\sigma_i X_i(X_i) + m) \prod_{j=1}^m \frac{\sigma_j (X_i(X_i) + \sigma_{ij}) + m}{(2m p_{ji}^0 + p_{ji}^2)(X_i - X_{ji1})(X_i - X_{ji2}) + i\varepsilon} \quad (3.3a)$$

with

$$X_{ji1} = \frac{2m p_{ji}^0 + p_{ji}^2 \sqrt{1 - \frac{4m^2}{p_{ji}^2}}}{2m p_{ji}^0 + p_{ji}^2} \quad (3.3b)$$

and

$$K_i(X_i) = X_i / K_i^0 = K_i^1(X_i) = \frac{2m X_i}{X_i^2 - 1}, \quad K_i^0 = W_i = \frac{m(X_i^2 + 1)}{X_i^2 - 1}$$

After partial fraction decomposition we are left with

$$L_m = \sum_{i=1}^m \left\{ \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) dX_i \sum_{j=1}^m \left(\frac{A_{ji1}}{X_i - X_{ji1}} + \frac{A_{ji2}}{X_i - X_{ji2}} \right) \right\}.$$

The possible poles at $X_i = \pm 1$ have been neglected in accordance with what we have learnt from our consideration of $K_1 \rightarrow \pm \infty$

The coefficients are determined by Lagrange's formula

$$A_{ji1} = \lim_{X \rightarrow X_{ji1}} \left\{ (X - X_{ji1}) \tilde{J}_i(X) \right\}$$

to be

$$A_{ji1} = \mp 2\pi i \frac{X_i W(X_i) \bar{W}(X_i)}{p_{ji}^2 \sqrt{1 - 4m^2/p_{ji}^2}} \prod_{k=1}^m \frac{\sigma_k (X_i + \sigma_{ki}) + m}{(K_i^0 + p_{ki}^0)^2 - (K_i^1 + p_{ki}^1)^2 - m^2 + i\varepsilon} \cdot \frac{1}{a_j} \frac{W(K_j) \bar{W}(K_j)}{K_j^0 - W_j} \\ \equiv \frac{\mp 2\pi i}{p_{ji}^2 \sqrt{1 - 4m^2/p_{ji}^2}} \cdot T_{ji1} \quad \begin{matrix} K_i^1 = W_i \\ K_i^0 = W_i \end{matrix}$$

with $W(u) = u(u)$ for $\sigma_i = 1$ and $W(u) = i v(u)$ for $\sigma_i = -1$

Already by the first integration K_i was set on mass-shell. Now also K_j is on shell and each of the coefficients A_{ji1} factorizes in a scalar function times a product of two tree graphs (Fig.5).

We denote the two mass-shell solutions of K_{i_1} to be $K_{i_1}^2 = m^2$, implying $K_{j_1} = K_{i_1}$. As the trees depend only on the in and out-going momenta K_i, K_j we observe the symmetry property

$$T_{ji1} = T_{ij2} = T_{ji}.$$

Therefore, in the integration we are left with contributions from the boundaries ± 1 . Inserting for X_{ji1} the values given by (3.3b) we arrive at the result (expressed in Mandelstam variables

$$S_{i_1 i_2} = (p_i + p_j)^2, \quad t_{i_1 i_2} = (p_i - p_j)^2$$

$$L_n = \sum_{\substack{i_1, i_2=1 \\ n+1}}^n T_{i_1 i_2} \frac{-1}{\sqrt{-S_{i_1 i_2} t_{i_1 i_2}}} \ln \left(\frac{\sqrt{-S_{i_1 i_2}} - \sqrt{t_{i_1 i_2}}}{\sqrt{-S_{i_1 i_2}} + \sqrt{t_{i_1 i_2}}} \right) \quad (3.4)$$

So far we have excluded the case $n = 2$ because it leads after the K_0 -integration to a logarithmically divergent expression. For (2→2)-particle scattering the sum of the two different graphs (Fig.6) with external momenta on shell may be explicitly computed /20/ and leads to the same result which we would have obtained by just formally applying (3.4). With respect to (n→n)-particle scattering amplitudes, involving a $n = 2$ loop integral, we remark that the inductive arguments of the previous section may readily be generalized to prove that the principal values of the propagators in the trees do not contribute. Therefore, the incoming loop momenta may again be considered to be on shell and the results of the explicit calculation demonstrate (3.4) to be also true if $n = 2$.

b) Results up to the One Loop Approximation

The structural results of the tree approximation (chapter II.c) - i.e. sets of incoming and outgoing momenta being equal, no pair creation, and factorization of the S-matrix - remain true in the one loop approximation.

For (2→2)-particle scattering the physical values of (3.4) are obtained by going with $s_{12} + i\epsilon$ ($s_{12} \geq 4m^2$ in the physical region) onto the real axis. Using (2.6), (2.10), (2.11b) and (3.4) the (2→2)-particle scattering phase up to order g^2 becomes

$$\delta_{s_1} = -\frac{g}{2} \left(\frac{-t_{12}}{s_{12}} \right)^{1/2} + \frac{g^2}{\pi} \frac{2m^2}{\sqrt{-t_{12}s_{12}}} \left(\frac{2m^2 \ln \left(\frac{\sqrt{-s_{12}} - \sqrt{t_{12}}}{\sqrt{-s_{12}} + \sqrt{t_{12}}} \right)}{\sqrt{-t_{12}s_{12}}} - 1 \right) + \mathcal{O}(g^3) \quad (3.5)$$

Applying the substitution rule (2.3) at the beginning of the calculations the particle, antiparticle phase is obtained along the same lines.

The factorization of the one loop contributions in terms of tree graphs immediately implies that the arguments which prove in the tree approximation that the sets of incoming and outgoing momenta are equal in any scattering process remain unaffected.

Finally, again, factorization of the higher scattering amplitudes in terms of two particle scattering according to formulae (2.13) and (2.14) occurs. The contribution of the connected tree graphs to the amplitude is of order g^{n-1} , and the (connected) one loop graphs contribute in the order g^n . Factorization up to the one loop approximation means that these contributions are contained in the product of two-particle scattering amplitudes with the scattering phase expanded up to $\mathcal{O}(g^2)$. There arise clearly also higher order terms (up to $g^{2(n-1)}$) from the product. They have to be neglected at this stage and they are supposed to match with certain contributions from higher order loop diagrams.

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Figure Captions

Figure 1: 1a) denotes a configuration for particle-particle scattering in the tree approximation,
 1b) the particle, antiparticle configuration obtained by the substitution (2.3)

Figure 2: A configuration of the graph which determines (3+1)-particle scattering

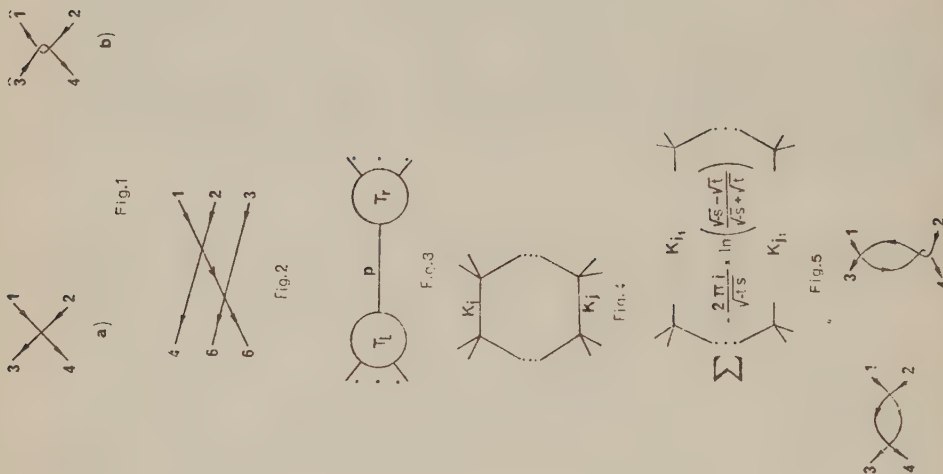
Figure 3: A propagator
$$p = \frac{i \sum_{\alpha} \frac{q_{\alpha}^2}{q_{\alpha}^2 - m^2 + i\epsilon} \cdot M(q_{\alpha}) \cdot \tilde{M}(q_{\alpha})}{q_{\alpha}^2 - m^2 + i\epsilon}$$

of a general configuration which contributes to $A_{n,n}^{\pm}$ is explicitly shown. T_L, T_r denote the general tree structures on the left, right respectively.

Figure 4: The general one loop integral

Figure 5: The factorization of the general loop integral

Figure 6: Configurations of the two graphs which contribute to two-particle scattering in the one loop approximation.



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THE BAXTER MODEL AND SCALE INVARIANT FIELD THEORY

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Abstract

We investigate the Baxter model and its special case, the two-dimensional Ising model, within the hypothesis of an underlying scale invariant field theory. The translation of the Euclidean lattice expressions for physical operators (spinors etc.) and pairs of the spin (order), disorder variables of the Ising model to corresponding expressions in terms of relativistic Majorana fields is carried out. All order, disorder correlation functions can be easily computed from the relativistic formulae. The generalization of such correlation functions to the Baxter model is obtained by switching on an energy-energy interaction between two Ising models. The energy-energy coupling turns out to be the Thirring coupling when linear combinations $\Psi = \psi_L + i\psi_R$ of Majorana fields are introduced. The critical indices $\alpha, \nu, \eta, \gamma$ and β are determined from the correlation functions and are seen to be consistent with the scaling picture.

I. INTRODUCTION

A commonly accepted working hypothesis based on the works of Wilson /1/ and Kadanoff /2/ states that critical phenomena (i.e. second order phase transitions) find their natural explanation in an underlying scale invariant field theory. However, this picture has been established with considerable rigour /3/ (for example A^4 in $D = 4$ dimensions) only in connection with mean field theory.

We investigate in detail the consequences of this hypothesis for the Baxter model /4/. In the language of ferromagnetism the Baxter model is equivalent to two superimposed (two-dimensional) Ising models interacting with one another via a four spin coupling term /5,6/. Naturally, we first consider the Ising model with nearest neighbour coupling. This part of the work was reported in a preprint by B. Schroer and the author /7/.

In the specific context of the Ising model the hypothesis of an underlying field theory means that at the critical point $T = T_c$ the long distance parts of the fluctuating (elementary or composite) variables define a globally scale invariant field theory in a Hilbert space. This "asymptotic" field theory has lost all memories of the lattice cutoff. Its natural setting is the local Euclidean field theory /8/ which may via analytic continuation be related to a local relativistic quantum field theory /9/. It has been argued /10/ that due to the mathematical pathology of Euclidean field operators of large dimensions ($\gg \frac{D}{2}$) it is convenient to study the relativistic field theory which is in the explained way affiliated with a critical theory. We carry out some steps of this program for the Ising model in section III.

Our basis is the fermion formulation of the Ising model given by Schultz, Mattis and Lieb /11/. In this approach the transfer matrix is fully diagonalized, and - as well as the σ -spin (order) variable expressed in terms of certain spinors C^x, C^y (called in the following SML-spinors). Additionally we introduce the disorder variable μ in terms of SML-spinors. The disorder variable μ plays an important role in the Kadanoff operator algebra /2,12/ of the Ising model.

This algebra (the relativistic formulation was developed independently and is known to be the Wilson expansion /13/) is based on the "reduction hypothesis" which states that a product of nearby fluctuating local variables can be replaced by a linear combination of individual local variables. For the two-dimensional Ising model Kadanoff /2/ has shown that it allows the determination of the critical indices from the knowledge of which coefficients in the algebra are non-vanishing. The crucial point is (besides certain special rules which are extracted from the properties of the lattice model) the hypothesis that all emerging operators are real physical operators obeying the known transformation properties of spinors, scalars, vectors, etc.

Our field theoretical method incorporates the Kadanoff algebra in the sense of that philosophy. The SML-fermion formulation of the lattice theory provides us with expressions on a line which we would like to translate into corresponding ones of a relativistic field theory. The known rotational properties of the operators allow us to generalize the two point functions on a special line to two point functions on an arbitrary line. The translation becomes especially straightforward because certain two point functions turn out to be those of free fields and, therefore, by Pohlmeyer's /14/ generalization of the Jost-Schroer theorem the corresponding fields themselves are free fields. We succeed in rewriting (for ordered pairs on a straight line) the lattice expressions for all σ_{ij} -correlations in terms of relativistic Majorana fields. From the relativistic formulae the result of Kadanoff and Ceva /12/ on the σ_{ij} -correlation functions for large distances is derived.

Section IV deals with the generalization of the results to the Baxter model. First, the Ising model results are reformulated for a doubled lattice of two non-interacting Ising models. The dynamic is now governed (in a non-trivial way) by the Dirac Hamiltonian /15/

$$H = \frac{i}{2} \int \bar{\Psi} \gamma^0 \partial_0 \Psi dx$$

of a massless Dirac field which is introduced in terms of Majorana fields corresponding to the sublattices. Only at this stage it is possible to obtain relativistic (path integral) expressions for the $(\mu-\mu)$ -, $(\epsilon-\epsilon)$ -two point functions with correct rotational properties in the Minkowskian region, i.e. yielding scalar rotational properties in the Euclidean region.

By switching on the four spin interaction between the Ising lattices we are led to the "dressed" formulae of the Baxter model. The relativistic equivalent of the four spin coupling turns out to be the Thirring /16/ interaction, and we can carry out the calculation of the higher order correlation functions by well known methods /17/. Thereby, according to our general philosophy, we treat the emerging potential as the potential of the real electrical current of the Thirring model which is normalized such that

$$Q = \int \mathcal{I}^0(x) dx$$

The precise result is a closed formula for all ψ_s, ψ_μ -correlations on a straight line; ψ_s and ψ_μ are electrical quantities defined in terms of operators belonging to the sublattices to be $\psi_s = \epsilon\epsilon'$ and $\psi_\mu = \mu\mu'$. From the correlation functions the critical indices $\alpha, \nu, \eta, \gamma$, and β are obtained. For vanishing four spin coupling the original Ising model indices are recovered (taking the half of the indices η, γ, β). All results are found in agreement with the generally assumed scaling laws.

Using different methods another field theoretical description of the Baxter model has been given by Luther and Peschel /18/. The results of both approaches are similar but not completely identical. Whereas the critical indices are the same there are deviations with respect to the higher ψ_s -correlations. The reason seems to be that Luther and Peschel use from the very beginning a "continuum generalization" of certain lattice models including a continuum generalization of the Jordan-Wigner transformation which obviously carries symmetry properties inexistent in the original lattice theory. On the other hand, the continuum generalization does not affect the critical properties and allows an operator formulation.

However, we believe in a field theory which is an exact limit of the lattice theory and, therefore, try a literal translation. In the limit of decoupled Ising models only our result for the spin correlations reduces exactly to Kadanoff's /19/ formula for the spin correlation functions in the Ising model.

With respect to the internal proofs we have tried to hold our material self-contained. The main sections are section III (Ising model) and section IV (Baxter model). Section II contains preparatory material from the lattice versions of the models. Some more technical details are deferred to appendices. Appendix A indicates a lattice calculation of the $\langle \mu \rangle$ -correlations, Appendix B contains the same calculation in terms of relativistic Majorana fields, and Appendix C collects bosonization formulae (known from the Thirring model) which are used in section IV.

II. PRELIMINARIES FROM THE LATTICE THEORIES

In this section we introduce the basic quantities in the lattice formulation. We start collecting formulae of the (two-dimensional) Ising model for later use. For the convenience of the reader we have adopted the notation of Schultz, Mattis, and Lieb /11/. Proofs may be found there.

The transfer matrix reads (magnetic field $H = 0$):

$$V = V_1 V_2 V_1$$

with

$$V_1 = (2 \sinh 2K_1)^{M/2} \exp \left[-K_1 \sum_{m=1}^M \sigma_m^z \right], \quad \tanh K_1 = \lambda^{-2K_1^*}$$

$$V_2 = \exp \left(K_2 \sum_{m=1}^M \sigma_m^x \sigma_{m+1}^x \right)$$

Following /11/ we introduce fermion operators (henceforth called SML-spinors) by a Jordan-Wigner transformation

$$C_m = \left[\exp \left(-\pi i \sum_1^{m-1} \sigma_i^+ \sigma_i^- \right) \right] \sigma_m^-, \quad C_m^+ = \left[\exp \left(-\pi i \sum_1^{m-1} \sigma_i^+ \sigma_i^- \right) \right] \sigma_m^+ \quad (2.1a)$$

with the inverse

$$\sigma_m^- = \left[\exp \left(-\pi i \sum_1^{m-1} C_i^+ C_i \right) \right] C_m, \quad \sigma_m^+ = \left[\exp \left(-\pi i \sum_1^{m-1} C_i^+ C_i \right) \right] C_m^+ \quad (2.1b)$$

The term in the sum is the energy density

$$\mathcal{E}_m = -C_m^+ C_m = \frac{1}{2} : C_m^x : C_m^y :$$

After the canonical transformation $\sigma^x \rightarrow \sigma^z, \sigma^z \rightarrow \sigma^x, \sigma^y \rightarrow -\sigma^y$ (and changing the direction of summation) these spinors can be closely related to Kadanoff's /20/ spinors b_{k+}, b_{k-} , explicitly

$$b_{k+} = \frac{1}{\sqrt{2}} C_{k-1}^x, \quad b_{k-} = \frac{-1}{\sqrt{2}} C_k^y$$

with

$$C_m^x = C_m^+ + C_m^-, \quad \text{and} \quad i C_m^y = C_m^+ - C_m^- \quad (2.2)$$

After introducing the Fourier decomposition

$$C_m = M^{-1/2} e^{-i\pi/4} \sum_q e^{iqm} \eta_q$$

the Bogoliubov-Valatin transformation

$$\xi_q = \cos \phi_q \cdot \eta_q + \sin \phi_q \cdot \eta_q^\dagger$$

$$\xi_q = \cos \phi_q \cdot \eta_{-q} - \sin \phi_q \cdot \eta_q^\dagger$$

diagonalizes the transfer matrix (written in fermion operators) to

$$V = (2 \sinh 2K_1)^{M/2} \exp \left[- \sum_{q, q' \in \Gamma} \varepsilon_q (\xi_q^\dagger \xi_{q'} - \frac{1}{2}) \right] \quad (2.3)$$

with

$$\cosh \varepsilon_q = \cosh 2K_1 \cosh 2K_2^* - \sinh 2K_1 \sinh 2K_2^* \cos q$$

In the Kadanoff operator algebra of the model /2,12/ the disorder variable μ plays a crucial role. We define it in terms of SML-spinors

$$\mu_m = \exp \left(-\pi i \sum_{i=1}^{m-1} C_i^\dagger C_i \right)$$

and obtain with (2.1) immediately (we use here and in the following the shorthand notation $\sigma \sim \sigma^x$)

$$\sigma_m = \mu_m C_m^x$$

$$C_m^x = \mu_m \sigma_m, \quad i C_m^y = \sigma_m^x \mu_{m+1}$$

Later the products of two σ, μ -variables become important. Their fermion formulation is

$$\mu_m \mu_{m'} = \exp \left(-\pi i \sum_{i=m}^{m'-1} C_i^\dagger C_i \right) \quad (2.4a)$$

$$\sigma_m \sigma_{m'} = i C_m^y \left[\exp \left(-\pi i \sum_{i=m}^{m'-1} C_i^\dagger C_i \right) \right] C_{m'}^x \quad (2.4b)$$

$$\sigma_m \mu_{m'} = i C_m^y \left[\exp \left(-\pi i \sum_{i=m}^{m'-1} C_i^\dagger C_i \right) \right] \quad (2.4c)$$

$$\mu_m \sigma_{m'} = \left[\exp \left(-\pi i \sum_{i=m}^{m'-1} C_i^\dagger C_i \right) \right] C_{m'}^x \quad (2.4d)$$

Contrarily to the continuous version the discrete version does not distinguish between $+\lambda\pi$ or $-\lambda\pi$ in the exponent. This is already true for the Jordan-Wigner transformation (2.1) and of importance for the question of an operator representation of the underlying field theory.

The long distance part of the higher order π/μ -correlations has been computed (nearly rigorously) by Kadanoff and Ceva /12/. Their result is the formula ($X_1, X_2, \dots, X_N$)

$$\left\langle \prod_{i=1}^N D_{\delta_i}(X_i) \right\rangle = \begin{cases} 0 & \text{if } \Gamma \neq 0 \\ \prod_{i,j=1}^N [c|X_i - X_j|]^{\delta_i \delta_j \mathcal{G}_i \mathcal{G}_j} & \end{cases} \quad (2.5)$$

Here:

$$\delta_i = \pm 1, \quad D_{\delta_i} = \sigma^x, \quad D_{-\delta_i} = \mu$$

$$\mathcal{G}_i = (-1)^{\Gamma_i}$$

and Γ is a quantum number defined recursively by

$$\Gamma_{i+1} = \Gamma_i + (-1)^{\delta_i} \delta_{i+1}, \quad \Gamma_i = \gamma_i, \quad \Gamma = \Gamma_N.$$

Although (2.5) is a formula for ordered pairs on a line it determines (using commutation of the operators) the general N -point function on a line as well. Based on (2.4) Appendix A contains a derivation of (2.5). The calculation is similar to the computation of the $(\psi-\psi)$ -correlation by T.T. Wu /21/.

The fermion operators $C^X(x)$ and $\lambda C^Y(x)$ behave (on a line) as free fields (higher point functions are sums of products of two point functions) with the two point functions

$$\langle C^X(x) C^X(x') \rangle = 0 = \langle \lambda C^Y(x) \lambda C^Y(x') \rangle \quad (2.6a)$$

$$\langle \lambda C^Y(x) C^X(x') \rangle = \frac{1}{\pi(x' - x + i/2)} \quad (2.6b)$$

Finally we review some results of the Baxter /4/ model. In the language of ferromagnetism the Baxter (eight vertex) lattice model consists of two superimposed Ising models coupled by an

additional four spin interaction /5,6/. The Hamiltonian reads

$$H = -kT \sum_{i,k} (K_1 \sigma_{i,k} \sigma_{i+1,k+1} + K_2 \sigma_{i,k} \sigma_{i,k+1} + \lambda \sigma_{i,k} \sigma_{i+1,k} \sigma_{i+1,k+1} \sigma_{i,k+1}) \quad (2.7)$$

There are exact results for the critical indices α /4/ and ν /22/ of the specific heat and correlation length respectively. These results exhibit a continuous variation of the indices with an interaction constant λ . Precisely

$$\sin\left(\frac{\pi\alpha}{4(1-\alpha_1)}\right) = \tanh 2\lambda \quad (2.8)$$

and the scaling law

$$\nu = 1 - \frac{1}{2}\alpha \quad (2.9)$$

For a field theoretical description we are particularly interested in the isotropic case of (2.7) $K = K_1 = K_2$. Since the relations (2.8), (2.9) are independent of K_1, K_2 the restriction $K_1 = K_2$ is not a serious one.

Correlation functions analog to (2.5) have not been computed for the Baxter model with lattice methods. Relying on field theoretical methods we obtain in section IV a generalization of (2.5) to the Baxter model.

Critical indices are determined by the correlation functions. Let

$$\xi = \left(\frac{T - T_c}{T_c} \right)^{-\nu}$$

be the correlation length, then the critical index α of the specific heat C_B and the critical index γ of the susceptibility χ are obtained from the integral formulae

$$C_B \sim \int_0^\xi x dx \langle E(x) E(0) \rangle \quad (2.10)$$

with $E = \sigma^2$ the energy density and

$$\chi \sim \int_0^\xi x dx \langle \sigma(x) \sigma(0) \rangle \quad (2.11)$$

For a review of this topic see Ref. 23.

III. THE ISING MODEL AT THE CRITICAL POINT IN FIFLD THEORETICAL TERMS

In this section we consider the Ising model of an isotropic lattice ($K_1 = K_2$) at the critical point $T = T_c$ (i.e. $K_1 = K_1^*$).

III.A) Hamiltonian Description

For $T \rightarrow T_c$ in the x-space the correlation length ξ approaches infinity and, consequently, in momentum space the dominant contribution comes from the region $q \approx 0$. Therefore, we replace ξ_q in formula (2.3) by $|q|$ the linear approximation in q . In dimensional units the range of summation is $\pm \frac{\pi}{q}$. Letting the lattice constant $a \rightarrow 0$, we go over to the integral formulation. The transfer matrix becomes proportional to

$$\int d^d q |q| \left(\xi_q^* \xi_q - \frac{1}{2} \right) = \mathcal{L}^{-H} \quad (3.1)$$

Introducing free massless Majorana fields ($\partial_\mu \gamma^\mu \psi = 0$ with $\gamma^0 = -i\sigma_2$, $\gamma^1 = -\sigma_3$ and σ_i the Pauli matrices) by

$$\psi_L(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ipx} a_L^+(p) dp + h.c. \quad u = t + x \quad (3.2a)$$

$$\psi_R(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ipx} a_R^+(p) dp - h.c. \quad v = t - x \quad (3.2b)$$

and doing the identifications $a_L(p) = \xi_p$, $a_R(p) = \xi_{-p}$

we recognize H to be the free Majorana Hamiltonian /7/

$$H = \frac{i}{2} \int \bar{\psi} \gamma^0 \partial_0 \psi \, dx \quad (3.3)$$

In the case of two non-interacting Ising systems on a doubled lattice, we are led to the introduction of a free Dirac field in terms of Majorana fields

$$\Psi = \psi_L + i \psi_R \quad (3.4)$$

and the Hamiltonian of the system is again given by equation (3.3), but with the Majorana field ψ replaced by the Dirac field Ψ . In this form the result was first obtained by Ferrell /15/. We conclude that the Ising model can be described (in a non-trivial way) as well

in terms of a free Majorana field as in terms of a free Dirac field if there is an underlying field theory at all.

Next we relate the SML-spinors introduced in the previous section to the Majorana fields. In consistence with (2.7) and the rotational properties of a spinor the two point function of

$$C(\vec{x}) = \begin{pmatrix} C^x(\vec{x}) \\ iC^y(\vec{x}) \end{pmatrix}$$

is

$$\langle C(\vec{x}) C(\vec{x}') \rangle = \frac{\sigma_3 \tau + \sigma_1 x}{\pi r^2}$$

Here we have adopted the notation (τ, x) for points in the Euclidean plane. Points in the two-dimensional Minkowskian space will be denoted by (t, x') . The two point function above is transformed from the Euclidean into the Minkowskian region by analytically continuing $\tau \rightarrow it$. Relating

$$C^x = \text{a.c.} \quad \psi_\lambda + i\psi_\tau \quad (3.5a)$$

$$iC^y = \text{a.c.} \quad i\psi_\lambda + \psi_\tau \quad (3.5b)$$

we obtain the Minkowskian two point function from the Majorana fields. Here and later the sign = a.c. indicates equality of Euclidean and Minkowskian operators in the sense that the Schwinger functions (n-point functions in the Euclidean region) of the first are identical with the analytically continued Wightman functions (n-point functions in the Minkowskian region) of the second, and vice versa.

The two point function given here is that of a free field, therefore, Pohlmeier's /14/ theorem ensures that there are only free fields with the same two point function. We remark that already the two point function (2.6) on a line and hermiticity conditions fix (3.5) to be the only possible relationship between the SML-spinors and the Majorana fields.

III.E) Disorder - Disorder Correlation

The main part of section III is devoted to the construction of the

ζ, μ -correlation functions. We begin with the $(\mu-\mu)$ -correlation. Using (3.5) a relativistic equivalent of (2.4a) is seen to be

$$\mu(\vec{x})\mu(\vec{x}') = \text{a.c.} \quad Z \left[e^{\lambda \int_{\vec{x}}^{\vec{x}'} E(\lambda) d\lambda} \right], \quad |\vec{x}| < |\vec{x}'| \quad (3.6)$$

The integral is understood along a straight line. Z is a renormalization factor to be defined later, and $E(\lambda) = \psi_\lambda(\lambda)\psi_\pi(\lambda)$ is the Minkowskian pendant of the Euclidean energy density $\mathcal{E}$. In the language of relativistic Lagrangian field theory E is seen to be a mass term and should not be confused with the zeroth component of the energy momentum tensor. For convenience, we keep the name of "energy density" as accepted in the nomenclature of statistical mechanics [2].

We require formulae in terms of (continuum) relativistic field operators which are analog to the lattice formulae (2.4). In a first step we have obtained (3.6) by formally translating the energy density $\mathcal{E}$ into its corresponding relativistic expression. The explicit calculation of the vacuum expectation value with (3.6) leads, however, to different results on different straight lines. This fact is connected with absence of relativistic invariance of the integral in the exponential of (3.6). Only the result on the time line (t -line) turns out to be correct.

The central clue for the calculation on the t -line is the possible bosonization of the energy operator on this line:

$$E(t) = \psi_\lambda(t)\psi_\pi(t) = E^{(+)}(t) + \text{h.c.}$$

with

$$E^{(+)}(t) = \frac{1}{2\pi} \int_0^\infty \lambda e^{\lambda t} c^\dagger(p) dp$$

and

$$[c(p), c^\dagger(p')] = p \delta(p-p')$$

The proof follows from the definition (3.2) by straightforward computation. Ordering the frequencies yields (all p 's positive)

$$c(p) = \int d\lambda \Theta(\lambda) [a_\pi(p-\lambda) a_\lambda(\lambda) \Theta(p-\lambda) - a_\lambda^\dagger(\lambda) a_\pi(\lambda+p) \Theta(p+\lambda) - a_\pi^\dagger(\lambda) a_\lambda(\lambda+p) \Theta(p+\lambda)]$$

We remark that the t -line is the only line where the bosonization of $E(t)$ is possible. There $E(t)$ has a potential

$$E(t) = \partial_t \varphi(t) \quad (3.8)$$

with (formally):

$$\varphi(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda}{\lambda - t} c^+(\lambda) d\lambda + \text{h.c.}$$

The potential is defined by its (infrared regularized) two point function

$$\langle \varphi(t) \varphi(t') \rangle = -\frac{1}{8\pi} \ln[-(t-t')^2 + \lambda \varepsilon |t-t'|]$$

and is treated on the basis of methods which are well known from the use of a scalar bose field in two dimensions /17,24/. With this in mind we define the renormalization factor Z such that

$$\mu(t)\mu(t') = \text{a.c.} \, Z \left[e^{\lambda \pi \int_t^{t'} \varphi(t'') dt''} \right] = : e^{-\lambda \pi \varphi(t)} : : e^{\lambda \pi \varphi(t')} : \equiv u(t)\mu(t') \quad (3.9a)$$

($t < t'$)

The resulting vacuum expectation value

$$\langle : e^{-\lambda \pi \varphi(t)} : : e^{\lambda \pi \varphi(t')} : \rangle = \frac{1}{[-(t-t')^2]^{1/8}}$$

yields by going to the euclidean region ($t \rightarrow -i\tau$)

$$\mu(\tau)\mu(\tau') = \frac{1}{|\tau - \tau'|^{1/4}}$$

which is the expected result. (Up to a proportionality constant which we have neglected in the relationship (3.9a).)

III.D) Higher Correlations

The higher σ, μ -correlation functions on a line are obtained by replacing (starting with the smallest τ) pairs of μ 's, σ 's by the relativistic expression (3.9a), (3.9b), and mixed pairs through the formulae

$$\sigma(\tau) \mu(\tau') = \text{a.c.} \lim_{\xi \rightarrow 0^+} (-\xi^2)^{1/4} (\psi_\tau(t-\xi) - \psi_\lambda(t-\xi)) \mu(\tau') \mu(\tau') \quad (3.9c)$$

$$\mu(\tau) \sigma(\tau') = \text{a.c.} \lim_{\xi' \rightarrow 0^+} (-\xi'^2)^{1/4} \mu(\tau) \mu(\tau') (\psi_\tau(\tau'+\xi') + \psi_\lambda(\tau'+\xi')) \quad (3.9d)$$

which (using (3.9a) and (3.11)) are translations of (2.4c) and (2.4d) in the previously explained sense. The ξ -renormalization factors are again uniquely determined.

These translations reproduce correctly all higher order correlation functions for an even number of σ, μ -operators on a straight line. Details of this calculation are left to Appendix B. For an odd number of σ, μ -operators the result is zero due to a single remaining operator. Therefore (2.5) is derived from our relativistic formulae. This is the most general consistence check which we have to offer for the correctness of our translation formulae.

To the extent that Wilson's heuristic arguments in favour of the A^4 -theory being in the same "universality class" (i.e. having the same scale invariant limit) as the Ising model are true in two dimensions, our σ -correlations refer to the (two-dimensional) massless A^4 -theory at the infrared fixed point.

III.C) Spin-Spin Correlation

As we are fixed to the t -line in the Minkowskian region we have, before doing the translation of formula (2.4b) into the Minkowskian region, to replace the operators ic^y, c^x which are emerging in (2.4b) by corresponding operators ic^y_z, c^x_z which have the two point function (2.6) not on the x but on the z -line. Going to the Minkowskian region we are looking for operators $(ic^y_z)_M, (c^x_z)_M$ fulfilling

$$\langle (ic^y_z)_M(t_1) (ic^y_z)_M(t_2) \rangle = 0 = \langle (c^x_z)_M(t_1) (c^x_z)_M(t_2) \rangle \quad (3.10a)$$

$$\langle (ic^y_z)_M(t_1) (c^x_z)_M(t_2) \rangle = \frac{1}{\pi} \frac{1}{(-t_1 - t_2)^{1/2}} \quad (3.10b)$$

In terms of Majorana fields the only possibility is seen to be

$$(ic^y_z)_M = \psi_\tau - \psi_\lambda \quad (3.11a)$$

$$(c^x_z)_M = \psi_\tau + \psi_\lambda \quad (3.11b)$$

By means of these relations we translate the lattice formula (2.4b) and arrive at the expression (again up to a proportionality constant)

$$G(z)G(z') = \text{a.c.} \lim_{\substack{\epsilon \rightarrow 0^+ \\ \epsilon' \rightarrow 0^+}} (-\epsilon)^{1/4} (-\epsilon')^{1/4} (\psi_\tau(t-\epsilon) - \psi_\lambda(t-\epsilon)) \mu(t) \mu(t') (\psi_\tau(t'+\epsilon') + \psi_\lambda(t'+\epsilon')) \quad (3.9b)$$

Here the uniquely determined ϵ -factors take care of the renormalization and make the limit well-defined. To calculate the vacuum expectation values the following commutation relations of the spinors with the $\phi^{(\pm)}$ -frequencies of potential are needed:

$$[(\psi_\tau(t) + \psi_\lambda(t)), \phi^{(\pm)}(t')] = \frac{\mp 1}{4\pi\lambda} \ln[-(t-t' \mp i\epsilon)^2] (\psi_\tau(t) + \psi_\lambda(t)) \quad (3.12)$$

The corresponding commutation relation for $\psi_\tau - \psi_\lambda$ follows by taking the hermitian adjoint. The final result

$$\langle G(z)G(z') \rangle = \frac{1}{|z-z'|^{1/2}}$$

reflects the Kramers-Wannier duality of order and disorder variable.

IV. FIELD THEORETICAL DESCRIPTION OF THE BAXTER MODEL

IV.A) Non-Interacting Ising Lattices

In the previous section we have derived formulae for the higher G, μ -correlations, which are valid on the t-line, in terms of Majorana fields. The first point of this section is the derivation of formulae which are valid on any straight line in the Minkowskian region.

For the (μ, μ) -correlation such a (path integral) formula has to be of the form

$$\mu(\tilde{x}) \mu(\tilde{x}') = \text{a.c.} \int \left[2 \pi \int_{(t, x')}^{(t', x'')} V_\mu ds^\mu \right]$$

i.e. the integrand being a vector or axial vector. But the operator $E(x) = \psi_1(x) \psi_2(x)$ is a scalar, and the Majorana vectors in question $\bar{\psi} \gamma^\mu \psi$, $\bar{\psi} \gamma^\mu \gamma_5 \psi$ are seen to be identically zero. We can, however, from the knowledge of the two point function on the t-line and the known transformation property construct the whole two point function. It comes out to be

$$\langle V_\mu(x) V_\nu(x') \rangle = -\frac{1}{8\pi^2} \partial_\mu \partial_\nu \ln \left[-(x-x')^2 + i\epsilon(t-t') \right]$$

The right-hand side is the two point function of a free vector field. Therefore, by Pohlmeyer's theorem V_μ itself is a free vector field. The current $j_\mu = \bar{\psi} \gamma_\mu \psi$ and the pseudocurrent $\tilde{j}_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi$ of the massless Dirac theory in two dimensions are known to be such free vector fields. According to the remarks following equation (3.4) the natural setting is the double lattice of two non-interacting Ising models. We introduce "electrical" operators by

$$d_\mu = \mu \mu' \quad \text{and} \quad d_\sigma = \sigma \sigma'$$

with μ, μ' ; σ, σ' the disorder, order parameters of the first (unprimed) and second (primed) lattice respectively. As a possible choice for the vector corresponding to the superimposed lattices we take $V_\mu + V_\mu' = j_\mu = \partial_\mu \phi$. Here ϕ is the substitute for a free field in two dimensions (cf. Appendix C), and the generalization of (3.9a) to a formula with correct rotational behaviour comes out

to be

$$\alpha_\mu(\vec{x}) \alpha_\mu(\vec{x}') = \text{a.c.} \quad : e^{-i\pi\phi(x)} : : e^{i\pi\phi(x')} : \quad (4.1a)$$

The corresponding two point functions are seen to be

$$\frac{1}{|\vec{x} - \vec{x}'|^{1/2}} = \text{a.c.} \quad \frac{1}{[-(t-t')^2 + (x'-x'')^2]^{1/4}}$$

Considering the two point functions of the free Dirac fields on the t-line we see the analog of formula (3.11) is

$$\begin{aligned} (\psi C_L^\gamma)_M &= \Psi_1, & (C_L^x)_M &= -\Psi_1^+ \\ (\psi C_E^\gamma)_M &= \Psi_2, & (C_E^x)_M &= -\Psi_2^+ \end{aligned}$$

Applying the bosonization formulae of Appendix C the free Dirac fields read (C 3), (C 4):

$$\Psi_1 = \frac{1}{\sqrt{\pi}} : e^{i\pi\phi_L(x)} : , \quad \Psi_2 = \frac{1}{\sqrt{\pi}} : e^{i\pi\phi_R(x)} :$$

and the ξ -limits analog to those in formulae (3.9b-d) can be carried out on the t-line. Afterwards the formulae, continued to an arbitrary line in consistence with the (scalar) $\sigma_{1/2}$ -transformation properties, are (up to a proportionality constant as in the previous section):

$$\alpha_\sigma(\vec{x}) \alpha_\sigma(\vec{x}') = \text{a.c.} \quad : e^{i\pi\phi(x)} : : e^{-i\pi\phi(x')} : \quad (4.1b)$$

$$\alpha_\sigma(\vec{x}) \alpha_\mu(\vec{x}') = \text{a.c.} \quad : e^{i\pi\phi(x)} : : e^{i\pi\phi(x')} : \quad (4.1c)$$

$$\alpha_\mu(\vec{x}) \alpha_\sigma(\vec{x}') = \text{a.c.} \quad : e^{-i\pi\phi(x)} : : e^{-i\pi\phi(x')} : \quad (4.1d)$$

Substituting again neighbouring pairs these formulae may be used for computing higher correlations on any straight line. Taking the square root one reproduces the $\sigma_{1/2}$ -correlations. We remark that the use of another free field (e.g. the pseudopotential $\tilde{\phi}$ instead of the potential ϕ) would require a different use of the spinor fields and then lead to the same result.

IV.B) Interacting Ising Lattices

Formulae (4.1) are the appropriate starting point for treating the Baxter model. The four-spin interaction (2.7) between the lattices amounts in adding to the free Dirac Lagrangian an energy density - energy density interaction

$$\mathcal{L}_{int} = 4g E_I E_{II} = \psi_{I\uparrow} \psi_{I\downarrow} \psi_{II\uparrow} \psi_{II\downarrow}$$

Using the relation (3.4) we write $\mathcal{L}_{int}$ in the Thirring /16/ form

$$\mathcal{L}_{int} = g : \bar{\Psi} \gamma_\mu \Psi \bar{\Psi} \gamma^\mu \Psi :$$

The equality reflects the fact that in two dimensions Fermi-statistics makes all quadrilinear interactions proportional.

According to Kadanoff and Wegner /6/ the scale invariant limit of the Baxter model can be directly obtained by a scale invariant Hamiltonian which, in our language of relativistic field theory, is the Hamiltonian of the Thirring model. More specifically, we are assuming that the operator identifications (4.1) of the Ising variables with the field theoretical expressions (on a straight line) remain unaffected.

According to the philosophy that we are dealing with real physical quantities, we require the potential ϕ to be the potential of the real electrical current of the Thirring model. The only thing what happens is that ϕ becomes dressed by turning on the interaction.

We now consider the interacting case in detail. The physical mass m has dimension 1 and is related with the scaling mass m_0 through $m \sim m_0^\nu$. We also know that the physical (interacting) Hamiltonian has dimension 1 (hence $\dim[\bar{\Psi} \Psi] = 2$, with Ψ the interacting massless Thirring field), and therefore:

$$\nu = (2 - \dim \bar{\Psi} \Psi)^{-1}$$

The critical index α for the specific heat is obtained from the energy-energy correlation via (2.10) to be

$$\alpha = \frac{2(1 - \dim \bar{\Psi} \Psi)}{2 - \dim \bar{\Psi} \Psi}$$

This is in accordance with the exact results of Baxter /4/ and Johnson, Krinsky, McCoy /22/ for α, ν respectively (2.8), (2.9). $\dim \bar{\Psi} \Psi$ may be related to the coupling constant, but this relation is of minor importance as it depends on the definition of the space time limit in the interaction term (cf. Appendix C). The expressions in terms of $\dim \bar{\Psi} \Psi$ are, however, universal; this means independent of any specific parametrization.

The most important point of this section is the computation of the higher α_s, α_n -correlations for the Baxter model. We first derive a relation between the free and the interacting electrical current. The real electrical current $\tilde{J}^\mu$ gives rise to the physical electrical charge by means of

$$Q = \int \tilde{J}^0(x) dx$$

This arises from the equal time commutator

$$[\tilde{J}^0(x), \Psi(x')]_{t=t'} = -\delta(x^1 - x'^1) \Psi(x) \quad (4.2)$$

with the interacting spinor field Ψ . Comparing this with the commutator (C.6) of the free current with the interacting field we are led to the multiplicative relation (C.7) between free and real current. The multiplicative factor is known as Schwinger term. The analog relation between free potential ϕ and interacting potential $\tilde{\phi}$ amounts (via (C.5)) to the replacement

$$\phi \rightarrow \tilde{\phi} = (\dim \bar{\Psi} \Psi)^{1/2} \phi \quad (4.3)$$

in the formulae (4.1) for the interacting case. The simple multiplicative dressing of ϕ enables us to calculate easily the interacting α_s, α_n -correlations along previously discussed lines (B.2). The two point function comes out to be

$$\langle \alpha_s(\vec{x}) \alpha_s(\vec{x}') \rangle = \langle \alpha_n(\vec{x}) \alpha_n(\vec{x}') \rangle = \frac{1}{|\vec{x} - \vec{x}'|^{1/2} \dim \bar{\Psi} \Psi} \quad (4.4)$$

The higher order correlations are seen to be of the structure (2.5); we only have to replace ϕ_i^μ by α_i, β_i respectively and the over-all exponents $\gamma_2 \gamma_3 \gamma_4$ by $2 \dim \bar{\Psi} \Psi \gamma_1 \gamma_4 \gamma_2 \gamma_3$.

For the pure α_i correlations there exists a result of Luther and Peschel /18/ which differs from ours by additional permutations. For example, to our four-point function $(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ on a straight line)

$$\langle \alpha_i(\vec{x}_1) \alpha_i(\vec{x}_2) \alpha_i(\vec{x}_3) \alpha_i(\vec{x}_4) \rangle = \left[\frac{|\vec{x}_1 - \vec{x}_3| |\vec{x}_2 - \vec{x}_4|}{|\vec{x}_1 - \vec{x}_2| |\vec{x}_1 - \vec{x}_3| |\vec{x}_3 - \vec{x}_4| |\vec{x}_2 - \vec{x}_4|} \right]^{\frac{1}{2} \dim \bar{\Psi} \Psi}$$

they would add the permutations $(2 \leftrightarrow 3)$ and $(1 \leftrightarrow 4)$.

Whereas we have translated ordered pairs of operators on a line, Luther and Peschel select a continuum version of the Jordan-Wigner transformation from the very beginning. The difference to our results seems to be due to a symmetry in this continuum version which is not incorporated in the lattice formulation. We believe in a field theory which reproduces the long distance behaviour of the lattice theory exactly. For the known Ising case /19/ this is afforded by our formulae on the price that we were not able to give an explicit operator formulation but only field theoretical ingredients for the translation on a line. We suspect an underlying field theory of a yet unsolved type which may reveal new interesting features.

With respect to the commonly discussed critical indices the results of both approaches are indistinguishable. The indices collected here correspond to the electrical interpretation of the model. From (4.4) we read off

$$\eta = \frac{1}{2} \dim \bar{\Psi} \Psi$$

According to (2.11) the index of the susceptibility is determined to be

$$\chi = \frac{2 - \frac{1}{2} \dim \bar{\Psi} \Psi}{2 - \dim \bar{\Psi} \Psi}$$

Finally the long-range order exponent β , defined by a mass-per-

turbation in $m_0 \sim \frac{T-T_c}{T_c}$ of the model, is determined from

$$\langle \psi(\vec{x}) \psi(0) \rangle \sim m^\eta, \quad m \sim m_0^\nu, \quad |\vec{x}| \rightarrow \infty \quad \text{to be}$$

$$\beta = \frac{\frac{1}{4} \dim \bar{\Psi} \Psi}{2 - \dim \bar{\Psi} \Psi}$$

This result is consistent with a plausible (non-exact) result of Baxter and Kelland /25/. Eliminating $\dim \bar{\Psi} \Psi$ we see the scaling laws

$$1 - \frac{1}{2} \alpha = \nu$$

$$\alpha + 2\beta + \gamma = 2$$

$$\gamma = (2 - \eta) \nu$$

being fulfilled.

V. CONCLUDING REMARKS

Quite naturally we have found field theoretical expressions which can be used to calculate higher order correlations in the Baxter model. The Ising model is included as a special case, and the results are in agreement with the previous lattice calculations /12/. Furthermore, all results were derived within the hypothesis of an underlying scale invariant field theory. Our conjecture is that the (Minkowskian) results on a straight line are the Wightman functions of a yet unsolved two dimensional field theory restricted to that line.

Interesting even for the Ising model is the generalization to a theory with mass (i.e. away from the critical point). Based on the mass perturbation theory of Coleman /26/, which exhibited the equivalence of the massive Thirring model and the Sine Gordon theory, some field theoretical results for small mass - in agreement with those following from the latest extensive lattice computations /27/ - were obtained by Itzykson and Zuber /28/.

There are several later investigations /29/ concerning the equivalence of the massive Thirring and the Sine Gordon theory. Along those lines it may be desirable to tackle the problem of the Ising (more generally Baxter) model away from the critical point. There is some hope /30/ that such an approach does not only lead to correlations away from the critical point on a straight line, but may also bring new insight to the question of the underlying field theory and may even lead to an explicit operator solution.

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Appendix A:

Lattice Computation of Higher Order Ising Model Correlation Functions

We indicate a proof of formula (2.5) with lattice methods. First we consider the special case of pure order correlations. Formula (2.5) simplifies to the result of Kadanoff /19/

$$\left\langle \prod_{\lambda \in A} (\sigma_{\lambda} \sigma_{\lambda'}) \right\rangle = \frac{\prod_{\lambda \in A} \prod_{\lambda' \in A} \langle \sigma_{\lambda} \sigma_{\lambda'} \rangle}{\prod_{\lambda \in A} \langle \sigma_{\lambda} \sigma_{\lambda} \rangle \langle \sigma_{\lambda'} \sigma_{\lambda'} \rangle} \quad (\text{A.1})$$

(with $\sigma_{\lambda} = \sigma(m_{\lambda})$, $\sigma_{\lambda'} = \sigma(m_{\lambda'})$, $m_{\lambda} = (m_{\lambda_1}, m_{\lambda_2}, \dots, m_{\lambda_n})$)

for all separations large.

Using (2.4b) we can substitute for each pair $\sigma_{\lambda} \sigma_{\lambda'}$ on the left-hand side SML-spinors

$$\sigma_{\lambda} \sigma_{\lambda'} = \lambda C_{m_{\lambda}}^Y C_{m_{\lambda'} + \lambda}^X \lambda C_{m_{\lambda} + \lambda'}^Y \dots C_{m_{\lambda'} - \lambda}^X \lambda C_{m_{\lambda} - \lambda'}^Y C_{m_{\lambda'}}^X \quad (\text{A.2})$$

In complete analogy to the spin-spin case treated by Schultz, Mattis, and Lieb /11/ the Wick-combinatorics of the spinors leads to the Toeplitz determinant

$$\left\langle \prod_{\lambda \in A} (\sigma_{\lambda} \sigma_{\lambda'}) \right\rangle = |t_k| = \begin{vmatrix} a_{m_1 m_1} \dots & a_{m_1(m_2-1)} & a_{m_1 m_2} \dots & a_{m_1(m_2-1)-1} & a_{m_1 m_2} \dots & a_{m_1(m_2-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(m_2-1)m_1} \dots & a_{(m_2-1)(m_2-1)} & a_{(m_2-1)m_2} \dots & a_{(m_2-1)(m_2-1)-1} & a_{(m_2-1)m_2} \dots & a_{(m_2-1)(m_2-1)} \\ a_{m_2 m_1} \dots & a_{m_2(m_2-1)} & a_{m_2 m_2} \dots & & & \\ \vdots & \vdots & \vdots & & & \\ a_{(m_2-1)(m_2-1)-1} m_1 \dots & & & & & \\ a_{m_2 m_1} \dots & & & & & \\ \vdots & & & & & \\ a_{(m_2-1)m_1} \dots & & & & & \\ & & & & & \vdots \\ & & & & & a_{(m_2-1)(m_2-1)-1} \end{vmatrix} \quad (\text{A.3})$$

with $a_{m m} = \langle \lambda C_{m_1}^Y C_{m_2}^X \lambda C_{m_1+1}^Y \dots \rangle$ and $1 \leq k \leq N = \sum_{\lambda \in A} (m_{\lambda_1} - m_{\lambda_n})$

The light lines only indicate the correspondence to the spin operators. If the distances between the spinors C^X, C^Y are large we have /21/

$$a_{mn} = \frac{1}{\pi(m-n+\frac{1}{2})} \quad (A.4)$$

We only calculate the Toeplitz determinant (A.2) with this approximation for a_{mn} . This result is known to be exact for the spin-spin correlation $\langle \sigma_{00} \sigma_{NN} \rangle$ diagonal in the lattice /31/. The corresponding result $\langle \sigma_{00} \sigma_{0N} \rangle$ for a horizontal or vertical line can be related to $\langle \sigma_{00} \sigma_{NN} \rangle$ by a very plausible (but mathematically non-rigorous) extension of Szegő's theorem /31/. For the isotropic lattice the result is in agreement with the physical picture of σ being a scalar.

Relying on these arguments we believe in the correctness of a computation of the (long distance part of the) spin correlation functions which uses the approximation (A.4) for a_{mn} .

With certain μ_k, ν_k the elements of the determinant (A.3) are of the form

$$t_{k\lambda} = (\mu_k + \nu_\lambda)^{-1}$$

Consequently (for a proof see e.g. /31/) the product structure

$$|t_{k\lambda}| = \frac{\prod_{k,\lambda} (\mu_k + \nu_\lambda) (\nu_k - \nu_\lambda)}{\prod_{k,\lambda} (\mu_k + \nu_\lambda)}$$

is obtained. Inserting the μ_k, ν_k explicitly the product is seen to be of the form

$$|t_{k\lambda}| = \frac{1}{(2\pi)^N} \frac{\prod_{m_1, m_2=1}^{m-1} [2(m-m_1)][2(m-m_2)]}{\prod_{m_1, m_2=1}^{m-1} [2(m-m_1)+1]} \quad (A.5)$$

with the regions $m_1, m_2 \leq m_{A+1}-1$ ($A=1, \dots, n$) omitted.

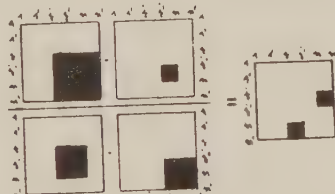
By drawing in a square the allowed regions (for the indices k, l) black and the not allowed ones white, we have a simple graphical equivalent of a product of this kind. In this notation:

$$\left\langle \prod_{k=1}^n (\sigma_k \tau_{k+1}) \right\rangle = |Z_n| =$$


formula (A.1) can now easily be proved by induction. It is clearly true for $n = 1$. We assume it to be true up to $(n-1)$ and rewrite (A.1) as

$$\left\langle \prod_{k=1}^n (\sigma_k \tau_{k+1}) \right\rangle = \prod_{k=1}^{n-1} \frac{\langle \sigma_k \tau_{k+1} \rangle \langle \sigma_{k+1} \tau_{k+2} \rangle}{\langle \sigma_k \tau_{k+1} \rangle \langle \sigma_{k+1} \tau_{k+2} \rangle} \langle \sigma_n \tau_{n+1} \rangle \cdot \left\langle \prod_{k=1}^{n-1} (\sigma_k \tau_{k+1}) \right\rangle$$

Graphically we see the identity

$$\frac{\langle \sigma_1 \tau_2 \rangle \langle \sigma_2 \tau_3 \rangle}{\langle \sigma_1 \tau_2 \rangle \langle \sigma_2 \tau_3 \rangle} =$$


and the proof (A.1) follows by using additionally the graphical notation for $\langle \sigma_n \tau_{n+1} \rangle$ and (by induction hypothesis) for

$$\left\langle \prod_{k=1}^{n-1} (\sigma_k \tau_{k+1}) \right\rangle.$$

Finally we indicate a straightforward but a little bit lengthy calculation for the generalization of (A.1) to (2.5). From (2.4) we obtain Fermion representations - in analogy to (A.2) - for all pairs of neighboured σ_i, τ_i -variables. The Γ -rule of (2.5) is immediately recognized to be equivalent to requiring the numbers of C^X and C^Y spinors being equal.

The case $\Gamma = 0$ is tedious. For the correlation function $\langle \prod_{\lambda=1}^n (D_{\lambda} D_{\lambda'}, 1) \rangle$ we obtain again a product formula which differs from (A.5) only by cancelled rows (columns). Now Wallis formula [32] can be used to calculate the quotient

$$\left\langle \prod_{\lambda=1}^n (D_{\lambda} D_{\lambda'}, 1) \right\rangle / \left\langle \prod_{\lambda=1}^n (\sigma_{\lambda} \sigma_{\lambda'}, 1) \right\rangle$$

By taking out pairs (i, i') and pairs $(i, i'), (k, k')$ of pairs of operators the whole calculation can be divided into twenty simple subcalculations (including the cases $D_{\lambda} = \sigma_{\lambda}, \lambda = \lambda', \lambda', k, k'$).

As example, one of the four calculations corresponding to pairs:

$$\frac{\langle \dots \sigma_{\lambda} \mu_{\lambda'} \dots \rangle}{\langle \dots \sigma_{\lambda} \sigma_{\lambda'} \dots \rangle} \text{ leads to } \frac{\prod_{\lambda=m_{\lambda}}^{m_{\lambda'}-1} [(2m_{\lambda'} - \lambda + 1) - (2\lambda)]}{\prod_{\lambda=m_{\lambda}}^{m_{\lambda'}-1} [(2m_{\lambda} - \lambda + 1) - (2\lambda + 1)]} \sim (m_{\lambda'} - m_{\lambda})^{1/2},$$

and one of the sixteen calculations corresponding to two pairs:

$$\frac{\langle \dots \sigma_{\lambda} \sigma_{\lambda'} \dots \sigma_{\mu} \mu_{\mu'} \dots \rangle}{\langle \dots \sigma_{\lambda} \sigma_{\lambda'} \dots \sigma_{\mu} \sigma_{\mu'} \dots \rangle} \text{ leads to } \frac{\prod_{\lambda=m_{\lambda}}^{m_{\lambda'}-1} [(2m_{\lambda'} - \lambda + 1) - (2\lambda)]}{\prod_{\lambda=m_{\lambda}}^{m_{\lambda'}-1} [(2m_{\mu} - \lambda + 1) - (2\lambda + 1)]} \sim \frac{(m_{\lambda'} - m_{\lambda})^{1/2}}{(m_{\mu'} - m_{\mu})^{1/2}}.$$

Collecting all results we obtain the desired expression (2.5).

Appendix B:Field Theoretical Computation of Higher Order Ising Model Correlation Functions

Details of the Majorana field calculation of formula (2.5) are given. An essential part of this appendix consists in establishing for Majorana fields on the time line results which are similar to known ones /17, cf. also appendix C/ for Dirac fields.

Let us first consider the special case of pure disorder correlations. From formula (3.9a) it follows

$$\left\langle \prod_{i=1}^{2n} \lambda(t_i) \right\rangle = \left\langle \prod_{i=1}^{2n} : \lambda^{(-1)^i} \lambda \varphi(t_i) : \right\rangle \quad (\text{B.1})$$

Using

$$: \lambda \varphi(t) : = \lambda \varphi^{(+)}(t) \lambda \varphi^{(-)}(t)$$

and the Baker-Hausdorf formula for c-number commutators

$$\lambda A \lambda B = \lambda B \lambda A + \lambda \lambda [A, B] \quad ([A, B] \text{ c-number})$$

we obtain

$$\prod_{i=1}^{2n} : \lambda_i \varphi(t_i) : = : \prod_{i=1}^{2n} \lambda_i \varphi(t_i) : \prod_{i,j} \lambda_i \lambda_j [\varphi^{(-)}(t_i), \varphi^{(+)}(t_j)] \quad (\text{B.2})$$

Using

$$[\varphi^{(-)}(t_i), \varphi^{(+)}(t_j)] = \langle \varphi(t_i) \varphi(t_j) \rangle = -\frac{1}{8\pi i} \ln[-(t_i - t_j) + i\epsilon(t_i - t_j)]$$

the right-hand side of (B.1) comes out to be (i, i' are odd and j, j' are even integers in the range $1 \dots 2n$)

$$\left[\frac{\prod_{i,j} [- (t_i - t_j)^2]}{\prod_{i,j} [- (t_i - t_{i'})^2] \prod_{j,j'} [- (t_j - t_{j'})^2]} \right]^{1/8}$$

By going to the euclidean region all higher disorder correlation functions are reproduced correctly.

For computing mixed order, disorder correlation functions we need the commutation relations of Majorana fields (3.2) ψ_x with the potential φ on the t-line.

The anticommutation relations of Majorana fields

$$\{\psi_x(\mu), \psi_x(\mu')\} = \delta(\mu - \mu')$$

$$\{\psi_r(v), \psi_r(v')\} = -\delta(v - v')$$

$$\{\psi_x(\mu), \psi_r(v)\} = 0$$

lead on the t-line to

$$[\psi_x(t), E(t')] = \delta(t - t') \psi_r(t) \quad (B.4a)$$

(with all indices - here and in the following - always in corresponding orders). (B.4a) is easily seen using the definition of the energy density $E(t) = \psi_x(t) \psi_r(t)$ and $[A, BC] = \{A, B\}C - B\{A, C\}$.

Linearity and unique frequency decomposition of the δ -function imply for the frequencies of the energy density (3.8a)

$$[\psi_x(t), E^{(\pm)}(t')] = \frac{\pm 1}{2\pi i (t - t' \mp i\epsilon)} \psi_r(t) \quad (B.4b)$$

This result can, of course, also be established by the (tedious) direct momentum space calculation. The corresponding commutation relations with the potential (3.8b) $\varphi(t)$ follow by integration.

$$[\psi_x(t), \varphi(t')] = -\Theta(t - t') \psi_r(t) \quad (B.5a)$$

$$[\psi_x(t), \varphi^{(\pm)}(t')] = \frac{\mp 1}{4\pi i} \ln[-(t - t')^2 \pm i\epsilon(t - t')] \psi_r(t) \quad (B.5b)$$

The last equation yields for the linear combinations $(\psi_r - \psi_x)$, $(\psi_r + \psi_x)$, (corresponding to the spinors χ_L^Y, χ_L^X respectively), the formula (3.12) and its hermitian adjoint.

The Minkowskian expression for the even (mixed) ξ, μ -correlation functions on a straight line is obtained from the translation formulae (3.9a-d). Now the strategy for the calculation is to move

all factors $\lambda^{\pm i\pi\varphi^{(+)}(t)}$ to the left and all factors $\lambda^{\pm i\pi\varphi^{(-)}(t)}$ to the right. In doing this we encounter the factor (B.3) and additional factors according to the commutation of the exponentials with the spinors. Using

$$[B, A] = \lambda B \quad \leadsto \quad B \lambda^A = \lambda^A B$$

the additional factors are seen to be of the type (t, t') :

$$(\psi_r - \psi_s)(t) \lambda^{\pm i\pi\varphi^{(+)}(t')} = [- (t - t')^2]^{\pm \gamma_4} \lambda^{\pm i\pi\varphi^{(+)}(t')} (\psi_r - \psi_s)(t) \quad (\text{B.7a})$$

$$(\psi_r + \psi_s)(t) \lambda^{\pm i\pi\varphi^{(+)}(t')} = [- (t - t')^2]^{\mp \gamma_4} \lambda^{\pm i\pi\varphi^{(+)}(t')} (\psi_r + \psi_s)(t) \quad (\text{B.7b})$$

and

$$\lambda^{\pm i\pi\varphi^{(-)}(t)} (\psi_r - \psi_s)(t') = [- (t - t')^2]^{\pm \gamma_4} (\psi_r - \psi_s)(t') \lambda^{\pm i\pi\varphi^{(-)}(t)} \quad (\text{B.7c})$$

$$\lambda^{\pm i\pi\varphi^{(-)}(t)} (\psi_r + \psi_s)(t') = [- (t - t')^2]^{\mp \gamma_4} (\psi_r + \psi_s)(t') \lambda^{\pm i\pi\varphi^{(-)}(t)} \quad (\text{B.7d})$$

Finally we are left with a factor corresponding to the vacuum expectation value of the spinors

$$\langle (\psi_r - \psi_s)(z_{\nu_1}) \dots (\psi_r - \psi_s)(z_{\nu_{n_1}}) (\psi_r + \psi_s)(z_{\mu_1}) \dots (\psi_r + \psi_s)(z_{\mu_{n_2}}) \rangle = \begin{cases} 0 & \text{if } n_1 \neq n_2 \\ \frac{1}{(-i\pi)^{n_1}} \frac{\prod_{i,j} (t_{\nu_i} - t_{\nu_j}) (t_{\mu_i} - t_{\mu_j})}{\prod_{i,j} (t_{\nu_i} - t_{\mu_j})} \end{cases} \quad (\text{B.8})$$

To get a non-zero result we need an equal number of $(\psi_r - \psi_s)$ and $(\psi_r + \psi_s)$ factors. This is identical with the Γ -quantum number rule in expression (2.6). And the non-zero results are now easily seen to be the $\phi_{\mu\nu}$ -correlations just given by this expression.

Appendix C

Bosonization Formulae from the Massless Thirring Model

In this appendix we collect formulae which may be used to write down a completely bosonized version of the massless Thirring model (including the free Dirac case). These formulae are fundamental for the calculations of section IV.

First we introduce left (l) and right (r) handed potentials

$$\varphi_l(\mu) = \frac{1}{4\pi\sqrt{\pi}} \int_0^\infty \frac{dp}{p} e^{ip\mu} c_l^+(p) + \text{h.c.} \quad \mu = t + x^1$$

$$\varphi_r(v) = \frac{1}{4\pi\sqrt{\pi}} \int_0^\infty \frac{dp}{p} e^{ipv} c_r^+(p) + \text{h.c.} \quad v = t - x^1$$

which are actually defined by their (regularized) two point functions

$$[\varphi_l(\mu), \varphi_l(\mu')] = \langle \varphi_l(\mu) \varphi_l(\mu') \rangle = -\frac{1}{4\pi i} \left\{ \ln(\mu - \mu' - i\epsilon) + \frac{i\pi}{2} \right\} \quad (\text{C.1a})$$

$$[\varphi_r(v), \varphi_r(v')] = \langle \varphi_r(v) \varphi_r(v') \rangle = -\frac{1}{4\pi i} \left\{ \ln(v - v' - i\epsilon) + \frac{i\pi}{2} \right\} \quad (\text{C.1b})$$

$$[\varphi_l(\mu), \varphi_r(v)] = \langle \varphi_l(\mu) \varphi_r(v) \rangle = 0. \quad (\text{C.1c})$$

The potential ϕ , pseudopotential $\tilde{\phi}$, of the free current $j_\mu = \frac{1}{\sqrt{\pi}} \partial_\mu \phi$ pseudocurrent $\tilde{j}_\mu = \frac{1}{\sqrt{\pi}} \partial_\mu \tilde{\phi}$, read in terms of φ_l, φ_r :

$$\phi = \varphi_l + \varphi_r \quad \text{and} \quad \tilde{\phi} = \varphi_l - \varphi_r \quad (\text{C.2})$$

Their two point functions are determined from (C.1) to be

$$\langle \phi(x) \phi(x') \rangle = \langle \tilde{\phi}(x) \tilde{\phi}(x') \rangle = -\frac{1}{4\pi i} \ln[-(x-x')^2 + i\epsilon(t-t')].$$

The completely bosonized solutions of the Thirring model are

$$\Psi_1(\mu, v) = \sqrt{\frac{1}{\pi}} : e^{i(\gamma_1 \varphi_l(\mu) + \gamma_2 \varphi_r(v))} : \sigma_1 \quad (\text{C.3a})$$

$$\Psi_2(\mu, v) = \sqrt{\frac{1}{\pi}} : e^{i(\gamma_2 \varphi_l(\mu) + \gamma_1 \varphi_r(v))} : \sigma_2 \quad (\text{C.3b})$$

The ξ_a are constant operators which take care of the Fermi structure

$$\langle \bar{\psi} \xi_a \bar{\psi} \xi_a \bar{\psi} \xi_a \bar{\psi} \xi_a \rangle = \delta_{m_1 m_2} \delta_{m_3 m_4}$$

The parameters γ_1, γ_2 are related to the dimension d_ψ and spin S_ψ of the interacting spinor field by means of

$$d_\psi - S_\psi = \frac{\gamma_1^2}{4\pi} \quad \text{and} \quad d_\psi + S_\psi = \frac{\gamma_2^2}{4\pi}$$

The canonical ($S_\psi = 1/2$) free Dirac case corresponds to $\gamma_2 = 2\pi$ and $\gamma_1 = 0$. For the interacting case the parameters may be related to the coupling constant g appearing in the interaction term $g \bar{\psi} \gamma_\mu \psi$. This relation, however, depends on what definition one uses for the interacting current $\tilde{J}^\mu$ as space time limit in spinor fields as is very clearly discussed in the paper of Dell'Antonio, Frishman, and Zwanziger /33/.

We are only interested in the limit of a massive Hamiltonian theory ($S_\psi = 1/2$), i.e. $\gamma_1^2 - \gamma_2^2 = 4\pi^2$. The choice $\tilde{J}^\mu = \frac{1}{i\pi} \dot{\psi}^\mu$ for the interacting current now would, in this case, lead to the relation $\gamma_1 = g \cdot c$. The special choice of c corresponding to the physical normalization (4.2) will be determined at the end of this appendix.

The bosonization may be used to compute the spectrum of the higher order composite fields. For the energy term $\bar{\psi} \psi$ the first non-trivial term in the Wilson expansion yields

$$\dim \bar{\psi} \psi = \frac{(\gamma_2 - \gamma_1)^2}{4\pi^2} \quad (C.5)$$

Finally we calculate with (B.6), (C.1), and (C.3) the commutator of the zeroth component of the free current $\dot{\psi}^0 = \partial^0 \psi$ with the interacting Thirring field $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

The result is

$$[\dot{\psi}^0(\mu, \nu), \psi(0)] = - \left(\frac{\gamma_1}{2\pi} \delta(\mu) + \frac{\gamma_2}{2\pi} \delta(\nu) \right) \psi(0) \quad (C.6)$$

The definition of the interacting current

$$\tilde{J}^0 \equiv \frac{2\pi}{\gamma_2 + \gamma_1} \dot{\psi}^0 = \frac{\gamma_1 - \gamma_2}{2\pi} \dot{\psi}^0 \quad (C.7)$$

gives the equal time commutation relation (4.2), and the interacting potential ($\tilde{J}_\mu = \frac{1}{\sqrt{\pi}} \partial_\mu \phi$) is related to ϕ by (4.3).

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